J. Fluid Mech. (1967), *vol.* 27, *part* 4, *pp.* 691–704
Printed in Great Britain

Laminar incompressible flow past a semi-infinite flat plate

By R. T. DAVIS

Virginia Polytechnic Institute, Blacksburg, Virginia

(Received 13 May 1966)

Laminar incompressible flow past a semi-infinite flat plate is examined by using the method of series truncation (or local similarity) on the full Navier–Stokes equations. The first and second truncations are calculated at points on the plate away from the leading edge, while only the first truncation is calculated at the leading edge. The solutions are compared with the results from other approximate methods.

1. Introduction

A new approach to the problem of the laminar flow past a semi-infinite flat plate is presented. Rather than the customary expansion procedures which are valid either near the leading edge or far downstream on the plate, an approximation scheme is developed based on the truncated series method or local similarity method. This method does not appear to be restricted to a low or high Reynolds number approximation. The difficulty of 'patching' the low Reynolds number (Stokes) approximation with the high Reynolds number (boundary-layer) approximation is avoided and a solution is obtained which is free from the undetermined constants characteristic of both the Stokes and boundary-layer approximations.

2. Governing equations and boundary conditions

Two co-ordinate systems naturally arise when one wishes to describe the flow past a flat plate. These co-ordinate systems are the rectangular Cartesian co-ordinate system and the parabolic co-ordinate system (see figure 1). We choose here the parabolic co-ordinate system since in this system the asymptotic solutions for the first approximation for both the low and high Reynolds number flows can be obtained merely by separation of variables, cf. Carrier & Lin (1948), Imai (1951), and Goldstein (1960). Rectangular Cartesian co-ordinates lead to considerable difficulty. For example, in Cartesian co-ordinates the Weiner–Hopf technique is required to solve the Oseen equations (Lewis & Carrier 1949) while the boundary-layer solution (Blasius 1908) must be obtained by finding a self-similar solution. Furthermore, the first-order boundary-layer solution in parabolic co-ordinates results in a uniformly valid solution to second order producing not only the proper form for the first-order inviscid flow in the outer region of the boundary layer but also the correct form for the flow due to displacement thickness. Rectangular Cartesian co-ordinates result in only the correct first-order inviscid solution. Co-ordinates which have the properties that the parabolic co-ordinates have in this problem are called optimal co-ordinates, see Kaplun (1954).

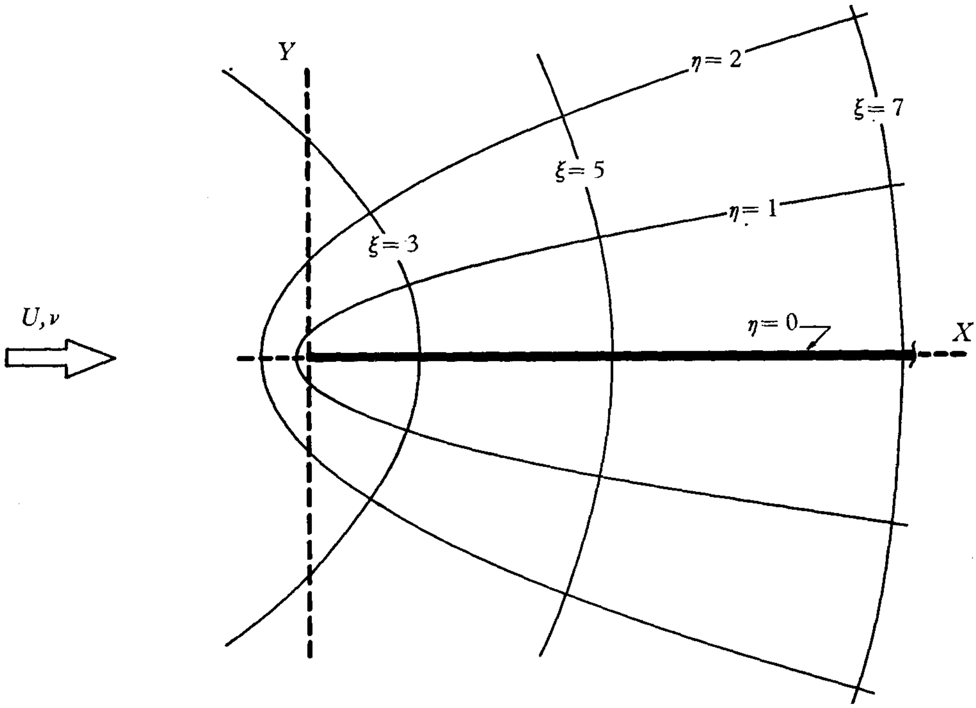

FIGURE 1. Co-ordinate system.

In non-dimensional parabolic co-ordinates the Navier–Stokes equations for plane flow are expressed in terms of the stream function as

$$[(\partial^2/\partial\xi^2 + \partial^2/\partial\eta^2) + \psi_\xi(\partial/\partial\eta) - \psi_\eta(\partial/\partial\xi)](\psi_{\xi\xi} + \psi_{\eta\eta})/(\xi^2+\eta^2) = 0, \tag{2.1}$$

where the stream function ψ is non-dimensionalized by the kinematic viscosity ν and

$$x + iy = \nu(\xi + i\eta)^2/2U, \tag{2.2a}$$

or

$$x = \nu(\xi^2 - \eta^2)/2U, \quad y = \nu\xi\eta/U, \tag{2.2b}$$

define the dimensionless parabolic co-ordinates.

With the flat plate situated along the positive x-axis with its leading edge at $x = 0$ (see figure 1) and the free stream velocity U assumed parallel to the plate, the appropriate boundary conditions become at the plate surface (assuming no slip)

$$\psi(\xi, 0) = 0, \quad \partial\psi(\xi, 0)/\partial\eta = 0, \tag{2.3a, b}$$

and at infinity

$$\psi(\xi, \eta) \sim \xi\eta \quad \text{as} \quad \eta \to \infty. \tag{2.3c}$$

3. Asymptotic approaches to the flat-plate problem

In the past most approaches to the flat-plate problem have consisted of finding asymptotic solutions to the Navier–Stokes equations valid either near the leading edge (Stokes flow) or far downstream from the leading edge (boundary-layer flow).

The solution for the Stokes flow near the leading edge has been obtained in polar co-ordinates by Carrier & Lin (1948) by neglecting the non-linear convective terms in the Navier–Stokes equations in the first approximation and then iterating to obtain higher approximations. Carrier & Lin's solution satisfying boundary conditions (2.3*a*, *b*) gives after correction (Van Dyke 1964*a*, p. 39)

$$\psi \sim \frac{A}{4} r^{\frac{3}{2}} \left(\cos\frac{\theta}{2} - \cos\frac{3\theta}{2}\right) + \frac{B}{12} r^{\frac{5}{2}} \left(\cos\frac{\theta}{2} - \cos\frac{5\theta}{2}\right) + \frac{A^2}{128} r^3 \{\sin^2\theta \times [\log r \sin\theta - (\theta-\pi)\cos\theta] + \tfrac{4}{5}(\sin 2\theta - 2\sin\theta) + C\sin^3\theta\} + O(r^{\frac{7}{2}}), \quad (3.1a)$$

where $\theta = 0$ on the plate surface, r is non-dimensionalized by ν/U, and the origin of the co-ordinate system is at the leading edge of the plate. In parabolic co-ordinates this solution (3.1*a*) becomes

$$\psi \sim \frac{A\sqrt{2}}{4}\xi\eta^2 - \frac{B\sqrt{2}}{24}[\xi\eta^4 - 3\xi^3\eta^2] + \frac{A^2}{64}[-\tfrac{2}{5}\xi\eta^5 + \tfrac{1}{2}(\xi^2\eta^4 - \xi^4\eta^2)\tan^{-1}(\xi/\eta) + \{\tfrac{1}{4}C - \tfrac{2}{5} + \tfrac{1}{2}\log\tfrac{1}{2}(\xi^2+\eta^2)\}\xi^3\eta^3] + O(\xi^\alpha\eta^\beta), \quad (3.1b)$$

where $\alpha + \beta = 7$.

The local coefficient of skin friction is obtained easily from (3.1*a*) or (3.1*b*) as

$$C_f = (\tau/\tfrac{1}{2}\rho U^2) \sim AR_x^{-\frac{1}{2}} + BR_x^{\frac{1}{2}} - (\pi A^2/32)R_x + \ldots, \quad (3.2)$$

where

$$R_x = Ux/\nu = \tfrac{1}{2}\xi^2. \quad (3.3)$$

We note that the skin friction will be singular or zero at the leading edge ($R_x = 0$) depending upon whether A in (3.2) is zero or finite.

Stokes paradox appears in the Stokes solution here as it does in other solutions for plane Stokes flows. We note that in the first approximation in (3.1*b*) the constant A is unknown and cannot be determined since the condition at infinity (2.3*c*) cannot be imposed. This difficulty was amended for the first approximation by Lewis & Carrier (1949) by using the Oseen equations in place of the Stokes equations. By doing this they were able to satisfy all of the boundary conditions (2.3*a–c*). Their solution is expressed in parabolic co-ordinates as

$$\psi = \xi\int_0^\eta \operatorname{erf}(N/\sqrt{2})\,dN, \quad (3.4)$$

with a resulting local coefficient of skin friction

$$C_f = (2/\sqrt{\pi})R_x^{-\frac{1}{2}} = 1{\cdot}13R_x^{-\frac{1}{2}}. \quad (3.5)$$

There is doubt, however, as to the correctness of this solution in describing flow near the leading edge of the plate. There is no doubt as to the validity of the Oseen equations for describing low Reynolds number flows over finite bodies; however, the validity of the Oseen equations for flows over semi-infinite plane bodies is in doubt. While one can say that, as the characteristic dimension of a finite body goes to zero, the body does not create a finite disturbance to the flow field, this is not true for a semi-infinite plane body. For a semi-infinite plane body (a parabola, for example), as the characteristic dimension of the body goes to zero (nose radius) the body still creates a finite disturbance due to the infinite surface area and resulting arresting action on the fluid. Because of this, various authors (see Lagerstrom 1964, pages 89 and 90, for example) have doubted the validity of the Oseen approximation for semi-infinite plane bodies.

Goldstein (1960) (see also Murray 1965) has obtained the asymptotic description of the Navier–Stokes equations (2.1) for large ξ. In the first approximation his equations correspond to the Blasius flat-plate equations. Higher approximations would lead to the description of the flow for moderate values of ξ to more accuracy except for the appearance of undetermined constants in these higher approximations. These constants presumably depend upon details of the flow near the leading edge of the plate. To the present time these remain undetermined. Goldstein's (1960) expansion (see Murray 1965 also) can be written in the following dimensionless form

$$\psi \sim \xi f_0(\eta) + \xi^{-1}[f_2(\eta) + g_2(\eta)\log\xi] + \ldots, \tag{3.6}$$

where

$$\xi = (2R_x)^{\frac{1}{2}}. \tag{3.7}$$

The function f_0 in the first approximation satisfies the familiar Blasius equation

$$f_0''' + f_0 f_0'' = 0, \tag{3.8}$$

with

$$f_0(0) = f_0'(0) = 0 \tag{3.9a, b}$$

and

$$f_0'(\eta) \sim 1 \quad \text{as} \quad \eta \to \infty. \tag{3.9c}$$

The equations for f_2, g_2 and higher approximations can be found in Murray (1965) with a slightly different choice of variables. The local coefficient of skin friction resulting from (3.6) is (Van Dyke 1964*a*, p. 140)

$$C_f \sim 0{\cdot}664 R_x^{-\frac{1}{2}} + (0{\cdot}551 \log R_x + C_1 - 1) R_x^{-\frac{3}{2}} + \ldots, \tag{3.10}$$

where C_1 is an undetermined constant.

Another very interesting approach to the flat-plate problem has been made by Dean (1954).† This is not an asymptotic approach in the sense of the Stokes and boundary-layer expansions, but instead makes the assumption that the first-order boundary-layer solution (equations (3.8) and (3.9)) is approximately valid everywhere and uses this solution to evaluate the non-linear convective terms in the Navier–Stokes equations in parabolic co-ordinates (2.1). This approximation, which seems irrational at first since boundary-layer theory is only good far downstream, turns out to be the best approach to the flat-plate problem to date. This is due to the fact that the solution in the whole flow field does not vary far from the first-order boundary-layer solution.

† The author is indebted to Milton Van Dyke for pointing out to him this interesting but often overlooked paper on flow past a flat plate. The author was unaware of Dean's work until after the present work on the flat plate was completed.

With the assumption that the non-linear convective terms can be approximated by first-order boundary-layer theory Dean finds the Green's function for a plane with a semi-infinite cut and reduces the solution to integrals which involve first-order boundary-layer quantities. In the variables used here his solution can be written as

$$\psi = \xi f_0(\eta) + \tfrac{1}{4}\xi\eta^2 \int_0^\infty \frac{F'(N)}{\xi^2+(2N+\eta)^2} N\,dN - \tfrac{1}{2}\xi \int_0^\infty \left[\frac{F(N+\eta)-F(N)-\eta F'(N)}{\xi^2+(2N+\eta)^2}\right] N^2 dN, \tag{3.11}$$

where

$$F(\eta) = (f_0 - \eta f_0')^2 - k^2, \tag{3.12}$$

f_0 being defined by (3.8) and (3.9) and k being defined by

$$k = \lim_{\eta\to\infty} (f_0 - \eta f_0') = -1{\cdot}21678. \tag{3.13}$$

Using this result, the local coefficient of skin friction is found to be

$$C_f = \sqrt{2}\, f_0''(0)\, R_x^{-\frac{1}{2}} + \frac{R_x^{-1}}{2\sqrt{2}} \int_0^\infty \left[\frac{F'(N)-NF''(N)}{R_x+2N^2}\right] N\,dN. \tag{3.14}$$

For large R_x this gives

$$C_f \sim 0{\cdot}664 R_x^{-\frac{1}{2}} + 3{\cdot}58 R_x^{-\frac{3}{2}} + \dots. \tag{3.15}$$

We note that the second term is not in agreement with (3.10).

A good check for all theories for flow over a flat plate is to find the integrated skin friction. Imai (1957) has found from momentum considerations that the integrated skin friction is given for large R_x by

$$C_F = \frac{\int_0^x \tau\,dx}{\frac{1}{2}\rho U^2 x} \sim 1{\cdot}328 R_x^{-\frac{1}{2}} + 2{\cdot}326 R_x^{-1} - (0{\cdot}204 + 2C_1 + 1{\cdot}102 \log R_x) R_x^{-\frac{3}{2}} + \dots, \tag{3.16}$$

where C_1 is the undetermined constant in (3.10). The second term, which is not obtained from formal integration of (3.10), takes into account the incorrectness of boundary-layer theory near the leading edge.

Any good method for treating the leading-edge problem should yield results which agree at least with the first two terms of equation (3.16). It is of particular interest to note that Dean's (1954) solution gives the correct values. (I am indebted to Milton Van Dyke for this remark.) This adds considerable weight to the validity of Dean's results and can be shown by integration of (3.14), which results in

$$C_F = 2\sqrt{2} f_0''(0)\, R_x^{-\frac{1}{2}} + \frac{R_x^{-1}}{2} \int_0^\infty \tan^{-1}\left(\frac{\sqrt{R_x}}{\sqrt{2}\,N}\right) [F'(N) - NF''(N)]\,dN. \tag{3.17}$$

For large R_x this gives

$$C_F \sim 1{\cdot}328 R_x^{-\frac{1}{2}} + 2{\cdot}326 R_x^{-1} - 7{\cdot}16 R_x^{-\frac{3}{2}} + \dots. \tag{3.18}$$

We note that the first two terms agree with (3.16), whereas the third does not.

Dean's results will be commented on later when they are compared with the results from the series truncation method.

4. Series truncation approach

Several problems in fluid mechanics have been solved by employing a method of series truncation primarily developed by Milton Van Dyke and co-workers. Solutions to several problems and a review of previous work in using the method are presented in several papers by Van Dyke (1964*b*, *c*, 1965*a*, *b*).

Basically the method consists of expanding the solution to a problem in a series in one of the independent variables. The dependence on the other independent variable or variables is left unspecified by letting the coefficients of the series be functions of those variables. The variable in which the expansion takes place and the manner in which the series is chosen are critical to obtaining an accurate solution with a few terms of the series. Also involved in obtaining a good solution is the choice of a proper co-ordinate system. The method appears to yield best results with a few terms if a co-ordinate system is used in the problem so that separation of variables almost works locally in the first approximation. The method has appeared to work best on problems involving a thin strip where the solution depends strongly upon local conditions on one or both sides of the strip and not on conditions far away from the region of interest. The best form for the truncated series in many of these cases is determined by the boundary conditions on the strip.

4.1. *Expansion about an arbitrary point*

Examining the forms of the solutions for the first-order boundary-layer (3.6), the first-order Stokes solution (3.1*b*) and the Oseen solution (3.4) for flow past a flat plate we see that all of these solutions take the form ξ times some function of η. This is rather unusual since the Stokes and Oseen solutions are for flow near the

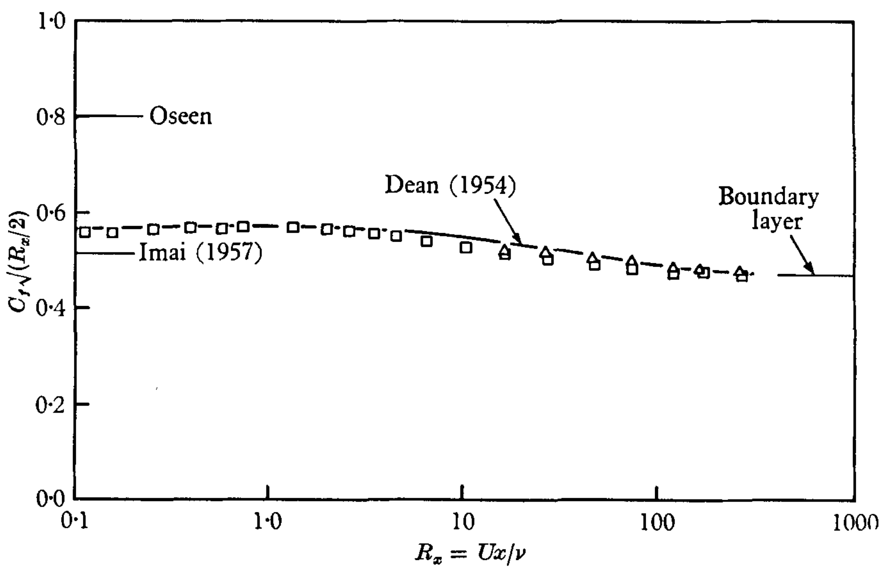

FIGURE 2. Local skin friction on a semi-infinite flat plate. □, first truncation; △, second truncation.

leading edge while the boundary-layer solution is for flow far downstream on the plate. Because of the form of the stream function in these two regions and the boundary condition at infinity it seems reasonable that for arbitrary ξ

$$\psi = \xi[g_1(\eta)+(\xi-\xi_0)g_2(\eta)+(\xi-\xi_0)^2g_3(\eta)+\ldots], \tag{4.1}$$

where ξ_0 is the position on the plate about which the expansion is taken and is equal to $(2R_x)^{\frac{1}{2}}$. Substituting the first two terms of (4.1) into (2.1) and equating

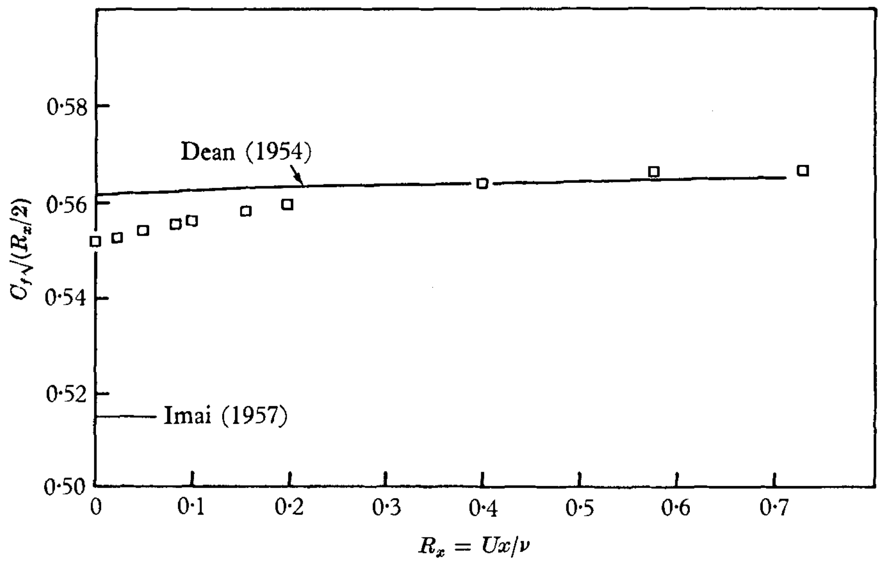

FIGURE 3. Local skin friction on a semi-infinite flat plate. □, first truncation.

coefficients of $(\xi-\xi_0)^0$ and $(\xi-\xi_0)^1$ to zero we obtain for the coefficients of $(\xi-\xi_0)^0$ the equation

$$g_1^{\text{iv}}+4g_2''+\frac{8g_2-8R_xg_2''-4\eta(g_1'''+2g_2')}{2R_x+\eta^2}+(g_1+2R_xg_2)$$
$$\times\left[2g_2'+g_1'''-\frac{2\eta}{2R_x+\eta^2}(2g_2+g_1'')\right]+g_1'\left[\frac{4R_x}{2R_x+\eta^2}(2g_2+g_1'')-g_1''-2R_xg_2''\right]=0, \tag{4.2a}$$

and for the coefficients of $(\xi-\xi_0)^1$ the equation

$$2R_xg_2^{\text{iv}}+\frac{6R_x+\eta^2}{2R_x+\eta^2}g_1^{\text{iv}}-\frac{4\eta}{2R_x+\eta^2}(g_1'''+2R_xg_2''')+2g_2\left[4R_xg_2'+2R_xg_1'''\right.$$
$$\left.-\frac{2\eta}{2R_x+\eta^2}(4R_xg_2+2R_xg_1'')\right]+(g_1+2R_xg_2)\left[g_1'''+2R_xg_2'''-\frac{2\eta}{2R_x+\eta^2}\right.$$
$$\left.\times(g_1''+2R_xg_2'')+\frac{2}{2R_x+\eta^2}(4R_xg_2'+2R_xg_1''')\right]+(g_1'+2R_xg_2')\left[\frac{2}{2R_x+\eta^2}\right.$$
$$\left.\times(4R_xg_2+2R_xg_1'')-g_1''-2R_xg_2''\right]+g_1'\left[\frac{8R_xg_2+4R_xg_1''}{2R_x+\eta^2}-4R_xg_2''\right]=0. \tag{4.2b}$$

The associated boundary conditions are from equations (2.3 *a*–*c*)

$$g_1(0) = g_1'(0) = 0, \tag{4.3a, b}$$

$$g_1(\eta) \sim \eta \quad \text{as} \quad \eta \to \infty \tag{4.3c}$$

and

$$g_2(0) = g_2'(0) = 0, \tag{4.3d, e}$$

$$g_2(\eta) \sim o(\eta) \quad \text{as} \quad \eta \to \infty. \tag{4.3f}$$

We note that both of equations (4.2 *a*) and (4.2 *b*) contain g_1, g_2 and their derivatives. Had we kept three terms, g_1, g_2, g_3 and their derivatives would appear in all three of the resulting equations. This is the usual occurrence with the series truncation method; however, if we have chosen our series in a good enough form we may be able to ignore the influence of the higher terms and obtain an accurate solution with a few terms.

As a first approximation assume that $g_2, g_3, \ldots, g_n$ and their derivatives have a small effect on the solution of (4.2 *a*). Ignoring those terms results in

$$g_1^{\text{iv}} - \frac{4\eta}{2R_x+\eta^2} g_1''' + g_1 g_1''' - \frac{2\eta}{2R_x+\eta^2} g_1 g_1'' + \frac{2R_x - \eta^2}{2R_x+\eta^2} g_1' g_1'' = 0 \tag{4.4}$$

with boundary conditions (4.3 *a*–*c*). This approximation (4.4) is called the first truncation.

The numerical solution of equation (4.4) can be simplified if one uses the fact that the Reynolds number can be removed from the differential equation by the transformation $g_1 \to (2R_x)^{-\frac{1}{2}} g_1$ and $\eta \to (2R_x)^{\frac{1}{2}}\eta$. The boundary conditions (4.3 *a*, *b*) are unchanged whereas (4.3 *c*) becomes $g_1(\eta) \sim 2R_x \eta$ as $\eta \to \infty$. The equation resulting from (4.4) can then be solved by giving values to $g_1''(0)$ and guessing $g_1'''(0)$ so that $g_1(\eta) \sim 2R_x \eta$ as $\eta \to \infty$ is satisfied. This determines a value of R_x associated with the choice of $g_1''(0)$. In this manner solutions are obtained by guessing only one initial condition; however one cannot determine the precise value of Reynolds number for the computation until the computation is completed.

Numerical solutions of (4.4) have been obtained for various values of R_x (i.e. ξ_0) in this manner. We apply the solution only at the point $\xi = \xi_0$ so that all of the higher terms in $\xi - \xi_0$ go out in (4.1) except in so far as they affect the solution for g_1 through the differential equations. These results for the first truncation appear in figures 2 and 3 for skin friction. One notes that, over the entire range of Reynolds number, the results do not vary far from the classical first-order boundary-layer results. One also notes that the results are amazingly close to those of Dean (1954) which can be obtained from (3.14). Both of these results indicate that the Oseen solution (3.5) is incorrect. The results of the first truncation suggest that Dean's assumption that the convective terms can be replaced by the boundary-layer values is a good one since the solution never varies far from the boundary-layer solution. The flat plate appears to be the only case where Dean's assumption will be valid, however, since other flows differ greatly for high and low Reynolds numbers. Both the results obtained here and those of Dean are in conflict with those of Kuo (1953), who finds the skin friction at the leading edge unchanged from the Blasius value.

The first truncation evaluated at $R_x = 0$† (see figure 3) can be used to evaluate the first constant in Carrier & Lin's (1948) Stokes solution (3.2). This results in a value of 0·779 for A, which is close to the value of 0·727 obtained by Imai (1957) by patching the Stokes and boundary-layer solutions together at $R_x = 1$, and is in excellent agreement with the value of 0·796 which can be obtained from Dean's (1954) results. The value of the undetermined constant C_1 appearing in (3.10) for the high Reynolds number flow can be computed from the first truncation; however, the first truncation is not accurate enough in the intermediate Reynolds number range to determine this constant to any degree of accuracy.

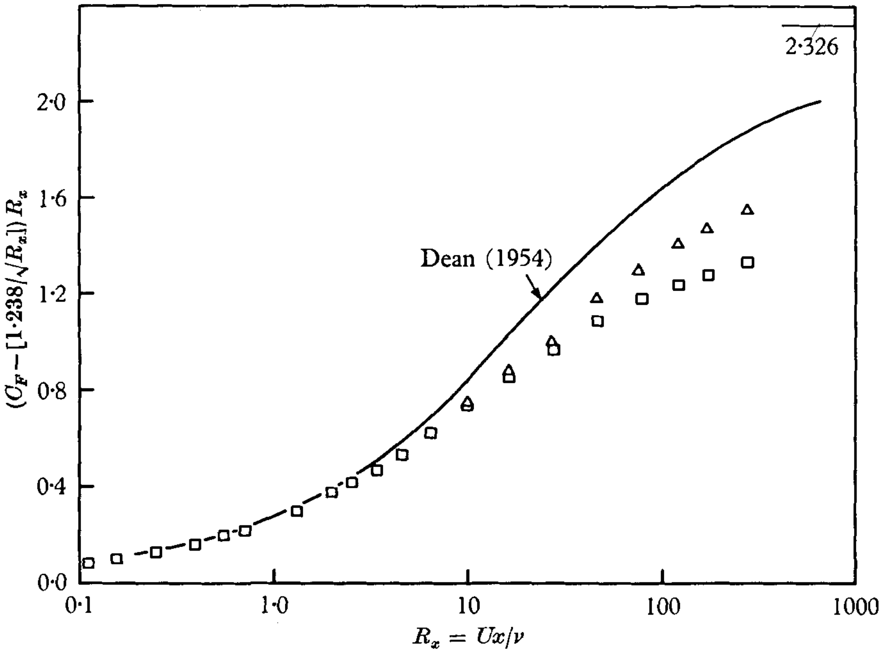

FIGURE 4. Integrated skin friction on a semi-infinite flat plate. □, first truncation; △, second truncation.

Figure 4 gives the results for the integrated skin friction obtained by using Simpson's rule for unequal step sizes. One sees that the first truncation appears to be too low to approach the value of 2·326 given by (3.16) as R_x goes to infinity. It was mentioned previously that Dean's (1954) results do approach this value and this can be seen from the figure; however Dean's results should not be relied on completely for the behaviour of the integrated skin friction since they do not have the correct form for the next higher approximation (see equation (3.18)).

The second truncation is obtained by including two terms in the series (4.1). This results in equations (4.2). These equations have been integrated numerically with the boundary conditions (4.3a–f) in the same manner as in the first truncation by using the similarity property of the equations. In this case, however, one must guess three initial conditions to satisfy the conditions at infinity. Newton's

† Difficulties encountered in the solution for this limiting case will be discussed later when the truncation at the leading edge is discussed.

method was used to interpolate to find the correct initial conditions. For Reynolds numbers above about 1, Newton's method converged and seemed to give acceptable results; however the low Reynolds number cases gave considerable difficulty in determining the proper initial conditions. Fortunately it was found that the second truncation did not change the results for the intermediate Reynolds number cases computed and in fact appeared to have little effect below Reynolds numbers of about 10 for the cases computed. One would expect that there would be some effect near Reynolds number of 0; however, those cases were difficult to compute and should be computed with a truncation centred at $R_x = 0$, which has a slightly different form than the one used here. The truncation at $R_x = 0$ will be discussed later.

Figures 2 and 4 give the results of the second truncation. In all cases the results move closer to the values given by Dean. The integrated skin friction, figure 4, is more acceptable and approaches more nearly the value of 2·326 as R_x goes to infinity. A third truncation would possibly supply the remaining difference.

With the results of the first two truncations one can evaluate the constant C_1 given in (3.10) and (3.16). Using the results for skin friction and (3.10) we obtain a value of about $C_1 = 0{\cdot}8$ for the second truncation, whereas the results for integrated skin friction along with (3.16) give a value of about $C_1 = 1{\cdot}5$ for the second truncation. These values should be in agreement with each other and should come closer as more terms are taken in the series (4.1). The corresponding values obtained from the first truncation are much further apart than the second truncation.

4.2. *Expansion at the leading edge*

In order to determine the nature of the flow near the leading edge of the plate we may choose to centre the expansion at $\xi = 0$ instead of at an arbitrary value of ξ. This leads one to assume an expansion of the form

$$\psi = \xi h_1(\eta) + \xi^3 h_2(\eta) + \dots. \tag{4.5}$$

One notices from (3.1 *b*) that the assumption of a power series expansion in ξ will not be appropriate beyond the second term in (4.5) due to the appearance of non-analytic terms in ξ in the third term of (3.1 *b*). The first two terms of (3.1 *b*) lead to

$$\psi \sim \left(\frac{A\sqrt{2}}{4}\eta^2 - \frac{B\sqrt{2}}{24}\eta^4 + \dots\right)\xi + \left(\frac{B\sqrt{2}}{8}\eta^2 + \dots\right)\xi^3 + \dots \tag{4.6}$$

which is consistent with (4.5).

Substituting (4.5) into the governing equation for the stream function (2.1) and equating coefficients of ξ^0 and ξ^2 to zero we obtain the following:

ξ^0,

$$\eta h_1^{\text{iv}} + 12\eta h_2'' - 4h_1''' - 24h_2' + 6\eta h_1 h_2' + \eta h_1 h_1''' - 12h_1 h_2 - 2h_1 h_1'' - 6\eta h_1' h_2 - \eta h_1' h_1'' = 0; \tag{4.7a}$$

ξ^2,

$$h_1^{\text{iv}} + \eta^2 h_2^{\text{iv}} - 4\eta h_2''' + 4h_2'' + 6h_1 h_2' + h_1 h_1''' + \eta^2 h_1 h_2''' - 2\eta h_1 h_2'' + 12\eta^2 h_2 h_2' + 3\eta^2 h_2 h_1''' - 36\eta h_2^2 - 6\eta h_2 h_1'' + 6h_1' h_2 + h_1' h_1'' - 3\eta^2 h_1' h_2'' - \eta^2 h_2' h_1'' = 0. \tag{4.7b}$$

The associated boundary conditions are

$$h_1(0) = h_1'(0) = 0, \tag{4.8a, b}$$

$$h_1(\eta) \sim \eta \quad \text{as} \quad \eta \to \infty \tag{4.8c}$$

and

$$h_2(0) = h_2'(0) = 0, \tag{4.9a, b}$$

$$h_2(\eta) \sim o(\eta) \quad \text{as} \quad \eta \to \infty. \tag{4.9c}$$

In addition we obtain from the differential equations (4.7a, b) and the boundary conditions (4.8a,b) and (4.9a,b) the conditions that

$$h_1'''(0) = 0 \tag{4.10}$$

and

$$h_1^{\text{iv}}(0) + 4h_2''(0) = 0. \tag{4.11}$$

By neglecting h_2 and its derivatives in equation (4.7a) we obtain the first truncation. This equation is exactly the same as (4.4) with $R_x = 0$. Thus we obtain for the first truncation

$$\eta h_1^{\text{iv}} - 4h_1''' + \eta h_1 h_1''' - 2h_1 h_1'' - \eta h_1' h_1'' = 0 \tag{4.12}$$

with the boundary conditions (4.8a,b,c).

In starting the numerical solution at $\eta = 0$ there is some difficulty due to the η appearing as a coefficient of the fourth derivative in the differential equation. One may overcome this difficulty and at the same time clarify some other points about the solution of (4.12) by finding the series solution to (4.12) satisfying boundary conditions (4.8a,b) for small η. Doing this we obtain

$$h_1(\eta) = \frac{A\sqrt{2}}{4}\eta^2 - \tfrac{1}{120}A^2\eta^5 + L\eta^7 - \frac{17A^3\sqrt{2}}{80{,}640}\eta^8 + \ldots, \tag{4.13}$$

where A and L are undetermined constants. The η^2 and η^7 terms arise from the series solution to (4.12) when the non-linear (convective) terms are neglected.

Using the series (4.13), the solution to (4.12) can be obtained by finding the values of A and L such that the boundary condition (4.8c) is satisfied. The numerical solution of (4.12) gives a value of 0·779 for A, as mentioned earlier, and a value of $0{\cdot}711 \times 10^{-3}$ for L. This solution is obtained easily if one notices that the differential equation (4.12) and the boundary conditions (4.8a,b) are invariant under the transformation

$$h_1 \to Kh_1, \quad \eta \to (1/K)\eta, \tag{4.14a, b}$$

thereby allowing one to find the solution by letting say $A = 1$ and then guessing values of L until $h_1''(\infty) = 0$. We then determine K such that $h_1'(\infty) = 1$ and thus determine A and L such that (4.8c) is satisfied. A similar similarity property was found to the Blasius equation (3.8) by Töpfer (1912).

One notices that (4.13) does not agree with (4.6) beyond the term in η^2. The term in η^4 in (4.6) does not appear in (4.13). We will find agreement in the η^4 terms in the second truncation; however, high-order terms in the Stokes solution not shown in (4.6) will not be in agreement with the series solution from the second truncation. Higher truncations appear to move the disagreement to higher terms in η.

The constant L appearing in (4.13) is apparently meaningless except that it allows us to determine a solution to the differential equation (4.13). In the next truncation the constant will appear in an even higher term in η, thus apparently having less and less effect on the solution.

For the second truncation we keep both terms in equations (4.7 a, b). We have the same difficulty with the second truncation as we had with the first at $\eta = 0$ and therefore must find the series solution for small η. Neglecting the non-linear convective terms in (4.7 a, b) one finds the series solution satisfying (4.8 a, b) and (4.9 a, b)

$$h_1 = \frac{A\sqrt{2}}{4}\eta^2 - \frac{B\sqrt{2}}{24}\eta^4 + \ldots + M\eta^{11} + \ldots, \tag{4.15}$$

$$h_2 = \frac{B\sqrt{2}}{8}\eta^2 + \frac{1}{36}A^2\eta^3\log\eta + N\eta^3 + \ldots - \frac{55M}{9}\eta^9 + \ldots. \tag{4.16}$$

The terms missing between the η^4 and η^{11} terms in (4.15) and the η^3 and η^9 terms in (4.16) arise from the non-linear terms in (4.7 a, b) and will involve the constants A, B and N. We note that (4.15) and (4.16) are in agreement with (4.6). The second term in (4.16) is the only term shown which is due to the non-linear terms.

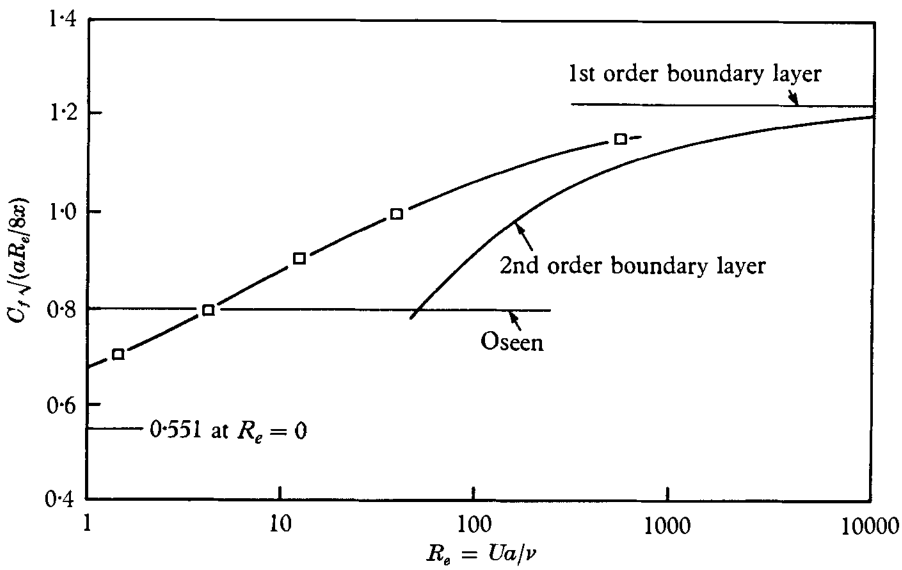

FIGURE 5. Skin friction at the stagnation point of a parabolic cylinder. □, first truncation.

We can now determine the solution to (4.7 a, b) since four constants are available (A, B, M and N) for satisfying the free-stream conditions (4.8 c) and (4.9 c). These solutions have not been found; however, work is being continued in this area. Presumably the value of A will change from the value found from the first truncation. Hopefully this change will be small.

Since parabolic co-ordinates are also useful in studying flows past parabolic-shaped bodies, we may use the equations derived for flow past a flat plate to also study the laminar viscous flow past a parabolic cylinder. Wang (1965) has applied equations (4.7 a, b), non-dimensionalized in a slightly different manner, to the flow near the stagnation point of a parabolic cylinder. The results for skin friction for the first truncation are shown in figure 5. Very little change was found when the second truncation was carried out. For comparison the results by Van Dyke (1964 d) for second-order boundary-layer theory are included along with the results of the solution of the Oseen equations by Wilkinson (1955). As in the case of the flat plate at low Reynolds number, it appears that the Oseen equations are not valid.

5. Concluding remarks

The series truncation method applied to flow past a semi-infinite flat plate appears to give reasonable results for all positions along the plate. This is due to the fact that the solution is slowly varying along the plate, never rising far above the results given by first-order boundary-layer theory. The fact that the solution never varies far from the first-order boundary-layer solution also explains the success of the method of solution used by Dean (1954).

The results obtained here for flow past a flat plate along with those obtained by Dean (1954) suggest that the Oseen equations are not applicable for flows past semi-infinite plane bodies. This is further substantiated by the results of Wang (1965) for flow near the stagnation-point of a parabolic cylinder.

In the series truncation method one finds that the equations for the first truncation for both the flow past a flat plate and the flow past a parabolic cylinder contain all of the terms which the Oseen equations contain for those flows plus some additional terms. If the Oseen equations are correct then the additional terms should be negligible in the solution. We have found that those additional terms do indeed influence the solution, thereby casting doubt on the validity of the Oseen equations. Higher truncations may change the results slightly; however, not of the magnitude needed to agree with the Oseen results.

This research was supported by NSF Contract no. GK-299.

REFERENCES

BLASIUS, H. 1908 Grenzschichten in Flüssigkeiten mit kleiner Reibung. *Z. Math. Phys.* **56**, 1.

CARRIER, G. F. & LIN, C. C. 1948 On the nature of the boundary-layer near the leading edge of a flat plate. *Quart. Appl. Math.* **6**, 63–68.

DEAN, W. R. 1954 On the steady motion of viscous liquid past a flat plate. *Mathematika* **1**, 143–156.

GOLDSTEIN, S. 1960 *Lectures on Fluid Mechanics*. New York: Wiley (Interscience).

IMAI, I. 1951 On the asymptotic behaviour of viscous fluid flow at a great distance from a cylindrical body, with special reference to Filon's paradox. *Proc. Roy. Soc.* A **208**, 487–516.

IMAI, I. 1957 Second approximation to the laminar boundary-layer flow over a flat plate. *J. Aeronaut. Sci.* **24**, 155–156.

KAPLUN, S. 1954 The role of coordinate systems in boundary-layer theory. *Z. Angew. Math. Phys.* **5**, 111–135.

Kuo, Y. H. 1953 On the flow of an incompressible viscous fluid past a flat plate at moderate Reynolds numbers. *J. Math. Phys.* **32**, 83–101.

Lagerstrom, P. A. 1964 Laminar flow theory. *Theory of Laminar Flows,* F. K. Moore ed. Princeton University Press.

Lewis, J. A. & Carrier, G. F. 1949 Some remarks on the flat plate boundary-layer. *Quart. Appl. Math.* **7**, 228–234.

Murray, J. D. 1965 Incompressible viscous flow past a semi-infinite flat plate. *J. Fluid Mech.* **21**, 337–344.

Töpfer, K. 1912 Bemerkung zu dem Aufsatz von H. Blasius 'Grenzschichten in Flüssigkeiten mit kleiner Reibung'. *Z. Math. Phys.* **60**, 397–398.

Van Dyke, M. 1964*a* *Perturbation Methods in Fluid Mechanics.* New York: Academic Press.

Van Dyke, M. 1964*b* The blunt-body problem revisited. *Proc. Internat. Hypersonics Symp.* 1964. Cornell University Press.

Van Dyke, M. 1964*c* The circle at low Reynolds number as a test of the method of series truncation. *Proceedings of 11th International Congress of Applied Mechanics.* Berlin: Springer Verlag, 1966.

Van Dyke, M. 1964*d* Higher approximations in boundary-layer theory. Part 3. Parabola in uniform stream. *J. Fluid Mech.* **19**, 145–159.

Van Dyke, M. 1965*a* Hypersonic flow behind a paraboloidal shock wave. *Journal de Mécanique,* December 1965.

Van Dyke, M. 1965*b* A method of series truncation applied to some problems in fluid mechanics. *VII Symposium on Advanced Problems and Methods in Fluid Dynamics.* Poland: Jurata 1965.

Wang, R. S. 1965 Laminar incompressible flow past a parabolic cylinder. Virginia Polytechnic Institute, M.S. Thesis.

Wilkinson, J. 1955 A note on the Oseen approximation for a paraboloid in a uniform stream parallel to its axis. *Quart. J. Mech. Appl. Math.* **8**, 415–421.